MATH 521A: Abstract Algebra Final Exam Fall 2016

Please read the following instructions. For the following exam you are allowed no books, notes, calculators, blank paper, or other materials. You have 120 minutes. Please write your answers in the designated spaces, and write legibly. To get credit, you must also show adequate work to justify your answers. If unsure, show the work. Each problem will be graded on a 5-10 scale, for a total score between 50 and 100.

Please turn in exactly ten problems, leaving the others blank: each of problems 1-6, and exactly four from problems 7-12.

1. Carefully define the term "quotient ring".

2. Carefully define the term "prime", as applied to ideals.

3. Carefully state the Division Algorithm Theorem (in $F[x]$).

4. Let $a, b \in \mathbb{Z}$. Prove that $\gcd(a, b) | \gcd(a + b, a - b)$, assuming that both numbers exist.

5. Find a ring homomorphism $f : \mathbb{Z} \to \mathbb{Z}[x]$, such that the image of f is not an ideal.

6. Find the minimal polynomial of $\sqrt{1+\sqrt{7}}$ over $\mathbb{Q}$.

Please solve exactly four of problems 7-12.

7. Let $p \in \mathbb{N}$ be irreducible. Prove that $p^4 + 14$ is reducible.

Please solve exactly four of problems 7-12.

8. Let R be a ring. We call $r \in R$ *idempotent* if $r^2 = r$. Suppose that R has 1, and let $x \in R$ be idempotent. Prove that $1 - x$ is idempotent.

Please solve exactly four of problems 7-12.

9. Let F be a field, and let $a, b \in F$. Prove that $\gcd(x^2 + a, x + b) = 1$ in $F[x]$, if and only if $a \neq -b^2$.

Please solve exactly four of problems 7-12.

10. Find the equivalence classes and rules for addition and multiplication in $\mathbb{Q}[x]/(x^2 - 4)$. Find all the units and zero divisors.

Please solve exactly four of problems 7-12.

11. Let F be a field, and let $f(x), g(x), h(x), p(x) \in F[x]$, with $p(x) \neq 0$. Prove that: $f(x)h(x) \equiv g(x)h(x) \pmod{p(x)}$, if and only if, $f(x) \equiv g(x) \pmod{\frac{p(x)}{\gcd(h(x),p(x))}}$.

Please solve exactly four of problems 7-12.

12. Define $I \subseteq \mathbb{Z}_3[x]$ via $I = \{f(x) : f(0)f(1) = 0\}$. Prove or disprove that I is an ideal in $\mathbb{Z}_3[x]$.